Using relaxational dynamics to reduce network congestion

Ana L. Pastore y Piontti,[1] Cristian E. La Rocca,[1] Zoltán Toroczkai,[2,3]
Lidia A. Braunstein,[1,4] Pablo A. Macri,[1] and Eduardo López[3]

[1]*Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Funes 3350, 7600 Mar del Plata, Argentina*

[2]*Department of Physics and Center for Complex Network Research, University of Notre Dame, 225 Nieuwland Science Hall, Notre Dame, IN, 46556*

[3]*Theoretical Division, Los Alamos National Laboratory, Mail Stop B258, Los Alamos, NM 87545 USA*

[4]*Center for polymer studies, Boston University, Boston, MA 02215, USA*

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Abstract

We study the effects of relaxational dynamics on congestion pressure in scale free networks by analyzing the properties of the corresponding gradient networks [1]. Using the Family model [2] from surface-growth physics as single-step load-balancing dynamics, we show that the congestion pressure considerably drops on scale-free networks when compared with the same dynamics on random graphs. This is due to a structural transition of the corresponding gradient network clusters, which self-organize such as to reduce the congestion pressure. This reduction is enhanced when lowering the value of the connectivity exponent λ towards 2.

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Transport networks, such as computer networks (Internet), airways, energy transportation networks, etc., are amongst the most vital components of modern day infrastructures. These large-scale networks have not been globally designed, instead they are the result of local processes. It has been observed that many networks have a scale-free (SF) connectivity structure [3, 4]. Scale-free networks are characterized by a power-law degree distribution $P(k) \sim k^{-\lambda}$, $(k \geq k_{min})$, where λ is the connectivity exponent and k_{min} is the lower degree that a node can have. There have been a number of mechanisms proposed in the form of stochastic network growth models that produce SF structures [3, 4] including weighted versions of these processes [5] (for discussions on the utility of SF models see Ref. [6]). However, these models do not *explicitly* connect the flow dynamics and transport performance (such as throughput, queuing characteristics, etc.) with network topology. This is difficult to do in general, since the time-scales of the flow on the network and that of the network's structural evolution itself can be rather different.

In this Letter we show that the emergence of scale-free structures is favored against non-scale-free structures, such as random graphs, if the transport dynamics has a relaxation component (called load-balancing in communications). We will see that *even one step* of such a gradient flow [1] will considerably reduce the congestion pressure in scale-free networks while it has virtually *no effect* in random graphs. In addition, within the class of uncorrelated scale-free networks, the congestion reduction is enhanced for low (close to 2) λ values. Although we use the jargon from the fields of communication networks and queuing theory, we expect that our results hold for large-scale networks in general, where the flow dynamics is induced at least in part by the existence of gradients, a rather ubiquitous mechanism. In the following, by "packet" we mean any discrete entity transported between two nodes of a network of N nodes. We assume that the network is driven in "the volume", by packets entering at random at an average rate γ at any of the nodes (this is realistic since the users actions in general are uncorrelated) [7, 8, 9, 10]. Using the language of queuing theory, if a node in the time interval $(t, t+\tau)$ sends packets into the network, but it receives no packets from any of its neighbors, we say that it acts as a "client", while if it receives a packet or several more from its neighbors, we say that it acts as a "server" [11]. Here $\tau > 0$ is the average processing time of a single packet by a node. Measuring the average fraction $J = \langle N_c/N \rangle$ of the number of clients N_c (over a period of time in the steady state), gives us a simple global measure for the congestion *pressure* present in the network [1]. The average

$\langle\cdot\rangle$ is over the randomness in the input but it can also be over network structure when comparing *classes* of networks. The client nodes are the ones that introduce new packets into the network, but they do not contribute to routing. Obviously, higher J means more congestion. Certainly, all networks will become congested at large enough driving rates $\gamma > \gamma_c$ [7, 8, 9, 10, 12, 13]. J indicates which network will become congested earlier, larger J meaning smaller γ_c. J is a global indicator that, however, does not take into account the *distribution* of the packets over the server nodes. That can be done via betweenness-based quantities as in Ref. [14]. Load-balancing is a specific case of the more general process of gradients induced flows [1], where the flows are produced by the local gradients of a non-degenerate scalar field $\mathbf{h} = \{h_i\}_{i=1}^N$ distributed over the N nodes of a substrate graph G (transport network). The scalar field could represent, for example, the number of packets at the routers [15, 16], or the virtual time horizon of the processors in parallel discrete event simulations [17]. The gradient direction of a node i is a directed edge pointing towards that neighbor (on G) j of i which has the lowest value of the scalar in i-s neighborhood. If i has the lowest value of h in its network neighborhood, the gradient link is a self-loop. The gradient network $\nabla_h G$ is defined simply as the collection of all gradient edges on the substrate graph G [1, 18]. It represents the subgraph of *instantaneous maximum flow* if the flow is induced by these gradients. In the gradient network each node has a unique outgoing link and ℓ incoming links. When a node has $\ell = 0$, (no inflow in that instant), it acts as a client, otherwise it is a server. Then certainly, J is the average fraction of nodes, with $\ell = 0$, i.e., it is the fraction of the "leaves" of $\nabla_h G$. Note that J is a queuing characteristic, rather than an actual throughput measure. It was shown [1, 18] that distributing random scalars $\{h\}$ independently onto the nodes of a network G, to which we refer as the *static* (S) model in the remainder, Erdős-Rényi (ER) graphs [19] (with given link probability p) become more congested with increasing network size N, i.e., $J_S \to 1$ while on SF networks J_S converges to a finite sub-unitary value, see the plots for J_S in Fig. 1. Once the (gradient) flow commences through the network, the scalar field becomes correlated and the queuing characteristics change. Usually, packets have destinations, and thus they cannot be governed exclusively by gradient flows, however, relaxational dynamics can be employed for finite periods of time. In the following we systematically study the effects of a *single* relaxational step, and show, that even in this case, the effects on congestion pressure can be drastic. First, we note that the one-step relaxation dynamics defined by the gradient flow is nothing but the

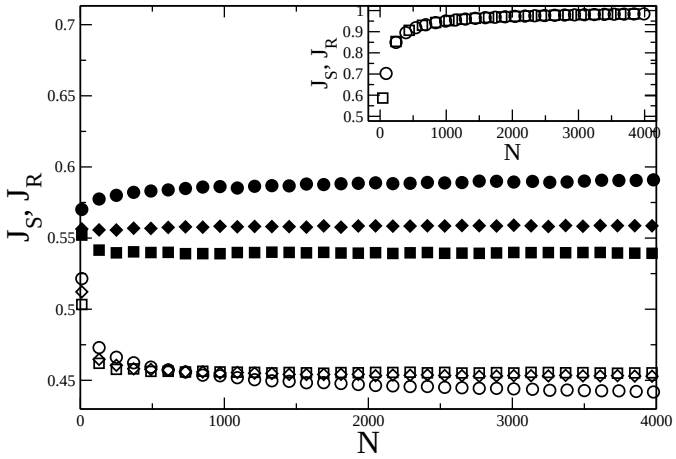

FIG. 1: J_S and J_R as function of N for SF networks with different values of λ, 2.5 ($\bigcirc$), 3($\diamond$), 3.5 ($\square$), with filled symbols for S and empty symbols for R. In the inset we plot J_S, ($\bigcirc$), and J_R, ($\square$), as function of N for ER networks ($p = 0.1$).

deposition model with surface relaxation (Family model) [2] from surface-growth physics extended to networks [20]. To generate SF networks, we used the configurational model [21] with $k_{min} = 2$ [22] (mostly for mathematical convenience, but see the discussion in the end about other networks). At $t = 0$ a random scalar field h is constructed by assigning to each node of the substrate network a random scalar independently and uniformly distributed between 0 and 1. At this stage the initial static gradient network [1] is formed and its jamming coefficient J_S determined. Then the scalars $h \equiv h(t)$ are evolved obeying the rules of the Family model [2]: at every time step a node i of the substrate is chosen at random with probability $1/N$ and it becomes a candidate for growth. If $h_i < h_j$ for every j (gradient criterion) which is a nearest neighbor of the node i, $h_i \to h_i + 1$. Otherwise, if h_i is not a minimum, the node j with minimum h is incremented by one. When the process reaches the steady state [20] of the evolution with this relaxation (R), we construct the gradient network and measure J_R. In accordance with previous observations [20, 23], the steady state is reached extremely fast: the saturation time actually does not scale with the system size N but it approaches an N-independent constant. In Fig. 1 we plot J_S and J_R as function of N for ER graphs and for SF networks (for the latter we compare cases with

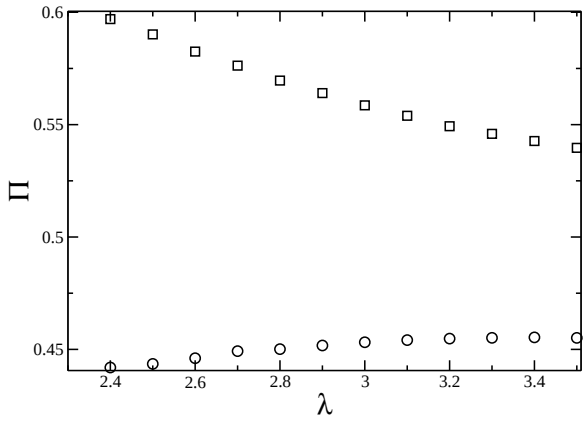

FIG. 2: Plot of Π, ◯ for R and □ for S, for SF networks as function of λ ($N = 4000$).

different λ values). As one can observe (inset), for ER networks, the model with relaxation has no effect on lowering the congestion, i.e., $J_S \simeq J_R$. For SF networks, however, there is a drastic difference between the static and dynamic cases, with J_S being considerably larger than J_R for large enough N. Note that J_S increases with decreasing λ, which can be understood through the fact that for lower λ values the $\nabla_h G$ of the SF graph is increasingly star-like, creating more congestion (see below). Since in real-world networks, however, one expects to find a load-balancing component of the transmission dynamics (see [15]), our model with relaxation is a better representation than the static one. And indeed, from Fig. 1 it becomes apparent that J_R has the *opposite* behavior as function of λ for large enough networks: lowering λ lowers the congestion pressure J_R. Next we show how the drop in the congestion pressure due to relaxation dynamics can be understood in terms of a structural change of the clusters of the corresponding gradient network. Here the clusters are defined as the disconnected components (trees) of the gradient graph. The decrease of J_R (which is the fraction of leaves of the gradient network) means a decrease of the perimeter of the clusters of $\nabla_h G$. To simplify the discussion, in the following we will use the symbol Π to denote the fraction of leaves (or the perimeter) of $\nabla_h G$, and thus $\Pi = J_{S\,or\,R}$. In Fig. 2 we show Π as function of λ for a fixed network size N. From this we can see that for a given value of N, Π is larger for the S model than the R one. This is compatible with a transition on the structure of the trees of the gradient network from a star-like structure in the S state to a more elongated structure in the R, see Fig. 3. This transition is responsible for the drop in the congestion pressure after the relaxation step is applied. From Fig. 3 we can see that for

the ER networks the R model does not affect significantly the structure of the clusters of the gradient network (going from Fig. 3a) to b)), meanwhile, for SF networks (going from Fig. 3c) to d)) we observe a major structural transition before and after the relaxation dynamics is applied, from a star-like cluster to an elongated one. For a quantitative insight *of the S*

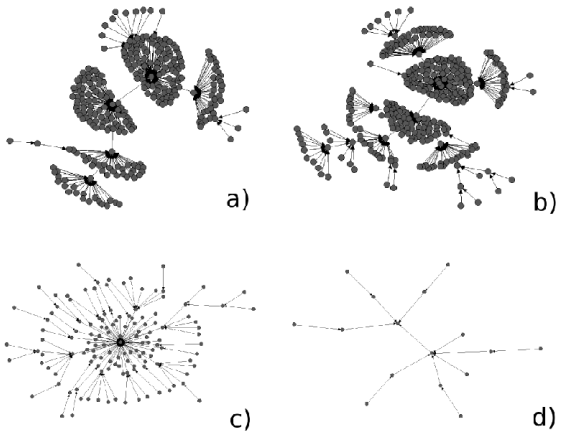

FIG. 3: Largest clusters of the gradient networks of ER and SF with $\lambda = 2.5$ and $N = 1024$. In the top plots, we show the structure for the ER and on the bottom for SF. The left plots (a),c)) correspond to the S and the right ones (b), d)) to the R models.

model consider that the scalars are identically (and independently) distributed according to some distribution $\eta(h)$. For the calculations below we assume that the SF network has low clustering, i.e., that the probability of two neighbors of a node to be also neighbors is small. Assume that the central node i has its scalar value equal to h_i. Then, the probability that the *neighbor* j of i points its gradient link (given h_i) into i equals to:

$$p_{j\to i}\Big|_{h_i} = \left[\int_{h_i}^{1} dh'\ \eta(h')\right]^{m_j} \equiv [\phi(h_i)]^{m_j}\ , \tag{1}$$

where $\phi(x) = \int_x^1 dh\ \eta(h)$ and m_j is the degree of node j. This is because the scalars at all the $m_j - 1$ neighbors of j (that do not include i) must be larger than h_i, and in addition we also have to have $h_j > h_i$. Hence the probability that node j *does not* point its gradient link into i is $(1 - p_{j\to i}|_{h_i})$. In order for node i to be a leaf on the gradient tree, one must have that none of its neighbors point their gradient directions into it. This is given by: $\prod_{j=1}^{k_i}(1 - p_{j\to i}|_{h_i})$, where for simplicity we labelled the neighbors of i by $j = 1, 2, \ldots, k_i$. Thus, the probability that i is a leaf is expressed by: $\pi_i = \int_0^1 dh_i\ \eta(h_i) \prod_{j=1}^{k_i}(1 - p_{j\to i}|_{h_i})$.

Using (1) and the change of variable from h to ϕ, $d\phi = -dh\ \eta(h)$, the integral becomes:

$$\pi_i = \pi_i(m_1, m_2, \ldots, m_{k_i}) = \int_0^1 d\phi \prod_{j=1}^{k_i} (1 - \phi^{m_j}) \ . \tag{2}$$

This expression shows that among all nodes i with the same degree k, those will likely

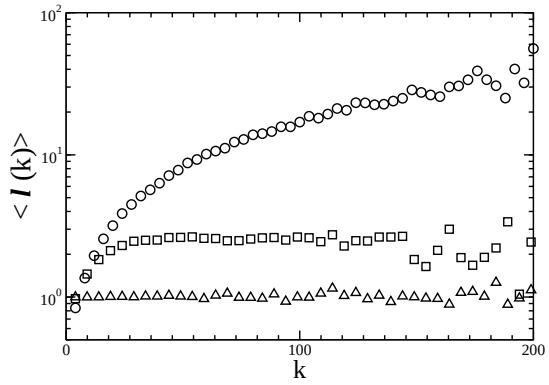

FIG. 4: Linear-log plot of $\langle \ell(k) \rangle$ as function of k with $\lambda = 2.5$ and $N = 1024$ for S (◯), and R (□), for $t = 0.1$ and R (△) in the steady state. As the system evolves, $\langle \ell(k) \rangle$ loses its dependence on k. We show the plot in the lin-log scale in order to be able to visualize them in the same figure.

be leaves in $\nabla_h G$, which have neighbors with *high degree*. It also shows that hubs will have very low probability of becoming leaves since in that case many (k) sub-unitary values are multiplied in Eq. (2). Therefore, leaves are coming from the set of nodes with low, or moderate degrees, connected to hubs, supporting Fig. 3c). When there is relaxation, the leaves are sending their load towards higher degree nodes (on average), correlating the scalars with the degree. This increases the chance of larger degree nodes to become leaves and breaking the star-like structures. As a consequence, the gradient network makes a transition to more elongated clusters, reducing its perimeter, and thus the congestion pressure. To support our claims numerically, we computed the average in-degree $\langle \ell(k) \rangle$ of $\nabla_h G$ as function of k in both the S and the R models. In the S model the gradient clusters are star-like, and thus they are very heterogeneous, which appears as a strong dependence of $\langle \ell(k) \rangle$ on k, see the circles in Fig. 4. After relaxation, however, this heterogeneity almost disappears (squares and triangles in Fig. 4), and in the steady state, $\langle \ell(k) \rangle \approx 1$ (only a weak k-dependence remains), which means that $\nabla_h G$ is made of elongated, homogenous trees. Defining the relative improvement factor as $F(N) = J_S/J_R$, which measures how efficient

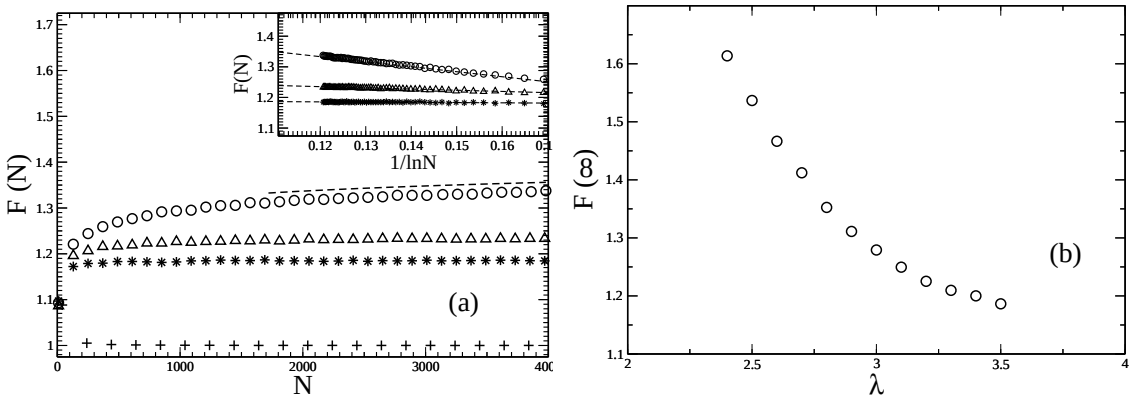

FIG. 5: a) $F(N)$ as function of N, for SF networks with $\lambda = 2.5$ ($\bigcirc$), 3.0 ($\triangle$), 3.5 ($*$) and for ER networks with $p = 0.1$ ($+$). The dashed lines correspond to the logarithmic fit of $F(N)$ as function of N in the asymptotic regime (see also inset). b) Asymptotic improvement factor as function of the connectivity exponent.

is the relaxation dynamics in reducing the congestion [24], we find that for SF networks, as λ decreases, $F(N)$ increases, see Fig. 5a), where we plot $F(N)$ as function of N for different values of λ. In addition, from Fig. 5a), we observe that $F(N)$ has a logarithmic convergence to its asymptotic value, i.e., $F(N) \simeq F(\infty) - K/\ln N$ (with K as a constant), see also the inset of Fig. 5a). Fig. 5b) shows the infinite-system size relative improvement factor as function of λ, showing the increasing power of the relaxation mechanism for lower λ values. The effects shown here seem to hold for other SF networks that we tested as well, including real-world networks such as the DIMES Internet mapping project generated AS level network (www.netdimes.org). The DIMES network is a correlated SF graph, and there the relaxation dynamics improves the congestion pressure drastically: from a $J_S = 0.61$ the congestion pressure drops to $J_R = 0.18$! This suggests that topology correlations play an important role which are the subject of present studies.

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